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Existence and uniqueness of nonnegative solutions for a boundary blow-up problem

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ABSTRACT

Assume that Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 3)$ with smooth boundary $\partial\Omega$. In this work, we study existence and uniqueness of blow-up solutions for the problem $-\Delta_p(u) + c(x)|\nabla u|^{p-1} + F(x,u) = 0$ in Ω, where $2 \leqslant p$. Under some conditions related to the function F, we give a sufficient condition for existence and nonexistence of nonnegative blow-up solutions. We study also the uniqueness of these solutions.

1. Introduction

In this paper, we investigate the existence and uniqueness of nonnegative weak solution for the following p-Laplacian problem

$$\begin{cases} -\Delta_p(u) + c(x)|\nabla u|^{p-1} + F(x,u) = 0 & \text{in } D'(\Omega), \\ u/\partial\Omega = \infty, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ and the operator $\Delta_p(u) = div(|\nabla u|^{p-2}\nabla u)$ with $p \geqslant 2$.

Problems like (1.1) are usually known in the literature as a boundary blow-up problems and its solutions named "blow-up solutions" or "explosive solutions" or "large solutions" of Eq. $(1.1)_1$. Precisely, by a solution of (1.1) we mean a positive solution of $(1.1)_1$ satisfying $u(x) \to \infty$, as $d(x,\partial\Omega) \to 0$. Notice that many works have been devoted to study this kind of problem. For $p = 2$, the first work introduced by Keller and Osserman is the following problem

$$\begin{cases} \Delta(u) = F(x,u) = g(x)f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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where $g = 1$ and $f : [0, +\infty) \to [0, +\infty)$ is a locally Lipschitz continuous function which is increasing. In [1] and [2] the authors proved that the problem (1.2) has a blow-up solution if and only if

$$\int^{\infty}\left(\int_0^s f(t)\,dt\right)^{-\frac{1}{2}} ds < \infty. \tag{1.3}$$

We also quote the paper [24], when $g \in C(\overline{\Omega})$ is nonnegative and satisfies the condition: For any $x_0 \in \Omega$,

$$\exists D \subset \overline{D} \subset \Omega \quad \text{such that} \quad x_0 \in D \quad \text{and} \quad g(x) > 0, \quad \forall x \in \partial D;$$

in this paper, Lair proved that the Keller–Osserman condition (1.3) remains still a necessary and sufficient condition for that the problem (1.2) admits a blow-up solution. In addition, when Ω is a ball and g is non-decreasing nonnegative radial weight, the problem was investigated by Bandle, Cheng and Porru in [6]. Recently, the existence for general bounded domain was obtained by Kim [20] but only for $f(u) = u^q$ and $q \in (1, \frac{N}{N-2})$, $N \geqslant 2$. In [25] and [26], Melián considered the questions of existence, boundary behavior and uniqueness of boundary blow-up solutions under the following condition: there exist positive constants $C_1, C_2 > 0$ and $\gamma_1, \gamma_2 > -2$ such that

$$C_1 d^{\gamma_2}(x) \leqslant g(x) < C_2 d^{\gamma_2}(x),$$

where $d(x) = d(x, \partial\Omega)$.

M. Ghergu and V. Rădulescu [21–23] studied the existence and nonexistence results for boundary blow-up solutions, which can be obtained to the elliptic equation

$$\Delta_p u + |\nabla u| = g(x) f(u) \quad \text{in } \Omega$$

and to the system

$$\begin{cases} \Delta u + |\nabla u| = g_1(x) f_1(v) & \text{in } \Omega, \\ \Delta v + |\nabla v| = g_2(x) f_2(u) & \text{in } \Omega, \end{cases}$$

where f, f_1 and f_2 satisfy a sub-linear growth condition at infinity.

Notice that important subjects were devoted to the question of existence and uniqueness, symmetry, convexity and asymptotic boundary behavior of blow-up solutions and led to several papers (see for example [3–6,12,27,28]). For $p \geqslant 2$, Mohammed [17,18] considered the problem

$$\begin{cases} \Delta_p(u) = F(x, u) = g(x) f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $f : [0, +\infty) \to [0, +\infty)$ is non-decreasing, $f(0) = 0$, $f(s) > 0$ for $s > 0$ and $g \in C(\overline{\Omega})$. The author proved that a sufficient condition for existence of solution of the problem (1.4) is the generalized Keller–Osserman condition:

$$\int^{\infty}\left(\int_0^s f(t)\,dt\right)^{-\frac{1}{p}} ds < \infty. \tag{1.5}$$

Recently, Mohammed [30], Huang and Tian [31] studied the asymptotic behavior of solution for the problem (1.4). We note that more blow-up problems have also been considered extensively by several authors (see [11,13–16,29,12] and the references therein).

Throughout this work we assume that $c \in L^\infty(\Omega)$ and F satisfies the following condition:

(f) For every $x \in \Omega$, $F(x, .) : [0, \infty) \to \mathbb{R}$ is non-decreasing and for any $\lambda > 0$,

$$\alpha(\lambda) := \inf_{u \geqslant 0}\big(F(x, u + \lambda) - F(x, u)\big) > 0.$$

Moreover, for every $u \geqslant 0$, $F(., u) : \Omega \to \mathbb{R}$ is continuous.

Remark 1.1. By a simple calculation we show that for every $a > 1$,

$$X^{p-1} - 1 \leqslant \left(\frac{a}{a-1}\right)^{p-2} (X-1)^{p-1} + a^{p-2} - 1, \quad \text{as soon as } X \geqslant a.$$

Then, $\frac{|\nabla u|}{|\nabla v|} \geqslant a > 1$ implies

$$\left||\nabla u|^{p-1} - |\nabla v|^{p-1}\right| \leqslant \left(\frac{a}{a-1}\right)^{p-2} |\nabla u - \nabla v|^{p-1} + \left(a^{p-2} - 1\right)|\nabla v|^{p-1}, \tag{1.6}$$

where $u, v \in W^{1,p}(\Omega)$.

This work is organized as follows: Section 2 – preliminary results. Section 3 – existence, nonexistence and uniqueness of blow-up solution. Section 4 – examples.

2. Preliminary results

We consider the following classical problem

$$\begin{cases} -\Delta(u) = 1, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

We know that the problem (2.1) has a solution $u_0 \in C^{2+\alpha}(\Omega) \cap C^2(\overline{\Omega})$, $\alpha > 0$. Let v_0 be a solution of the following problem

$$\begin{cases} -\Delta_p(v_0) = 1, & x \in \Omega, \\ v_0 = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

We take $H = \nabla u_0 \in C^\alpha(\overline{\Omega})$, then $div(|\nabla v_0|^{p-2}\nabla v_0 - H) = 0$. Moreover, by Guedda and Véron [8] v_0 is bounded. Then $v_0 \in C^{1,\alpha}(\overline{\Omega})$ (see [9]) and strong principle maximum gives $v_0 > 0$ in Ω.

We introduce some useful notation. Let $p > 1$ and let $W^{1,p}(\Omega)$ be the usual Sobolev space. We denote by $\|.\|_p$ the norm of $L^p(\Omega)$ and by $\|.\|_{\infty,\Omega}$ the norm of $L^\infty(\Omega)$. Define

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } p < \infty, \\ 1 & \text{if } p = \infty, \end{cases} \qquad p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \geqslant N, \end{cases}$$

and

$$\lambda_\infty = \sup_{x\in\Omega}\left\{2(p-1)v_0|\nabla v_0|^p + (p-1)|\nabla v_0|^p + v_0^2 + |c|v_0^2|\nabla v_0|^{p-1}\right\}. \tag{2.3}$$

Consider the problem

$$\begin{cases} \Delta_p(u) = h(x, u, \nabla u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.4}$$

Now, we give the result of existence of weak solutions of the problem (2.4), which is due to M.C. Leon (see [19, Theorem 2.2]).

Theorem 2.1. *We assume the following conditions*:

(H_1) *there exists an ordered pair $\alpha \leqslant \beta$ of sub- and super-solution of* (2.4).
(H_2) $|h(x, s, \eta)| < K(x) + a|\eta|^r$, $x \in \Omega$, $\forall s$: $\alpha(x) \leqslant s \leqslant \beta(x)$ *and* $\forall \eta \in \mathbb{R}^N$ *where*

$$K \in L^t(\Omega), \quad t > \left(p^*\right)' \quad \text{and} \quad 0 \leqslant r \leqslant \frac{p}{(p^*)'}.$$

Then the problem (2.4) *has at least one solution in* $W_0^{1,p}(\Omega)$ *between* $\alpha(x)$ *and* $\beta(x)$.

Another result that we will need is the interior regularity for weak solution to quasi-linear equation. It is due to DiBenedetto and Tolksdorf [7,10].

Theorem 2.2. *Suppose* $h(x, t, \eta)$ *is a measurable in* $x \in \Omega$ *and continuous in* t *and* η *such that* $|h(x, t, \eta)| < \Gamma(1 + |\eta|)^{p-1}$ *on* $\Omega \times \mathbb{R} \times \mathbb{R}^N$. *Let* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution of* $\Delta_p(u) = h(x, u, \nabla u)$. *For every* $y \in \Omega$ *and for every ball* $B(y, R)$ *in* Ω, *with* $R \in (0, 1)$, *there are an* $\alpha > 0$ *and a constant* C *depending only on* n, p, R, Γ *and* $\|u\|_{\infty,\Omega}$ *such that*

$$\begin{aligned} &\left|\nabla u(x) - \nabla u\left(x'\right)\right| < C\left|x - x'\right|^\alpha, \\ &|\nabla u| < C, \end{aligned} \tag{2.5}$$

for any $x, x' \in B(y, R)$.

3. Existence, nonexistence and uniqueness of blow-up solution

The main purpose of this section is to introduce some sufficient conditions for existence, nonexistence and uniqueness of blow-up solutions. First, we start with the maximum principle result which we will need later. Let $L : W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$ be the operator defined by

$$L(u) = -\Delta_p(u) + c(x)|\nabla u|^{p-1} + F(x,u).$$

We say that the maximum principle holds for L, if for every $u, v \in W^{1,p}(\Omega)$ such that $L(u) \geqslant L(v)$, in Ω and $u \geqslant v$ on $\partial\Omega$, we have $u \geqslant v$ in Ω.

Theorem 3.1 *(Maximum principle). Let $u, v \in W^{1,p}(\Omega)$ satisfy the respective inequalities*

$$\begin{cases} L(v) \leqslant L(u) & \text{in } \Omega, \\ v \leqslant u & \text{on } \partial\Omega. \end{cases}$$

If $|\nabla u|$ and $|\nabla v| \in L^\infty_{loc}(\Omega)$, then $v \leqslant u$ in Ω.

Proof. We suppose that $|\{x \in \Omega, v(x) - u(x) > 0\}| \neq 0$. For $n \in \mathbb{N}^*$, we define

$$\Omega_n = \{x \in \Omega / v(x) - u(x) > \lambda_n\},$$

where

$$\lambda_n = \begin{cases} \lambda - \frac{1}{n} & \text{if } \lambda < \infty, \\ n & \text{if } \lambda = \infty \end{cases}$$

with $\lambda = \operatorname{esssup}_{x\in\Omega}\{v(x) - u(x)\}$. Set

$$\Omega'_n = \{x \in \Omega / \nabla v \neq \nabla u,\ v(x) - u(x) > \lambda_n\},$$
$$G_{(a)} := \{x \in \Omega'_n,\ |\nabla u| \geqslant a|\nabla v|\},$$
$$\widetilde{G}_{(a)} := \{x \in \Omega'_n,\ |\nabla v| \geqslant a|\nabla u|\}$$

and

$$L_{(a)} := \left\{x \in \Omega'_n,\ a|\nabla u| > |\nabla v| > \frac{1}{a}|\nabla u|\right\},$$

where $a > 1$. It is easy to see that

$$w(x) = (v - u - \lambda_n)^+(x) = \sup\{0, v(x) - u(x) - \lambda_n\} \in W_0^p(\Omega).$$

Using (1.6), we obtain that

$$\begin{aligned} \int_{\Omega_n} \bigl|c(x)|\nabla u|^{p-1} - c(x)|\nabla v|^{p-1}\bigr| w &= \int_{\Omega'_n} |c|\bigl||\nabla u|^{p-1} - |\nabla v|^{p-1}\bigr| w \\ &\leqslant \|c\|_{\infty,\Omega}\left(\frac{a}{a-1}\right)^{p-2}\left(\int_{\widetilde{G}_{(a)}} |\nabla w|^{p-1} w + \int_{G_{(a)}} |\nabla w|^{p-1} w\right) \\ &\quad + (a^{p-2} - 1)\|c\|_{\infty,\Omega}\left(\int_{\widetilde{G}_{(a)}} |\nabla u|^{p-1} w + \int_{G_{(a)}} |\nabla v|^{p-1} w\right) \\ &\quad + \|c\|_{\infty,\Omega} \int_{L(a)} \bigl||\nabla u|^{p-1} - |\nabla v|^{p-1}\bigr| w. \end{aligned}$$

Note that $a|\nabla u| > |\nabla v| > \frac{1}{a}|\nabla u|$ implies that

$$\bigl||\nabla u|^{p-1} - |\nabla v|^{p-1}\bigr| \leqslant (a^{p-1} - 1)|\nabla v|^{p-1}.$$

Hence

$$\int_{\Omega_n'} \left| c(x)|\nabla u|^{p-1} - c(x)|\nabla v|^{p-1} \right| w \leqslant \|c\|_{\infty,\Omega} \left(\frac{a}{a-1} \right)^{p-2} \int_{\Omega_n'} |\nabla w|^{p-1} w + \left(a^{p-2}-1\right) \|c\|_{\infty,\Omega} \int_{\Omega_n'} |\nabla u|^{p-1} w$$
$$+ \left(a^{p-2}+a^{p-1}-2\right) \|c\|_{\infty,\Omega} \int_{\Omega_n'} |\nabla v|^{p-1} w. \tag{3.1}$$

Fix $n_0 \in \mathbb{N}$. From $|\nabla u|, |\nabla v| \in L^\infty_{loc}(\Omega)$, $\overline{\Omega_{n_0}'} \subset \Omega$ and $\alpha(\lambda_{n_0}) > 0$, we obtain that there is $a_0 > 1$ such that

$$\Phi := \left(a_0^{p-2}+a_0^{p-1}-2\right) \|c\|_{\infty,\Omega} |\nabla v|^{p-1} - \alpha(\lambda_{n_0}) + \left(a_0^{p-2}-1\right) \|c\|_{\infty,\Omega} |\nabla u|^{p-1} \leqslant 0 \tag{3.2}$$

a.e. in Ω_{n_0}'. Let $n > n_0$. Using the fact that $\langle L(u) - L(v), w \rangle \geqslant 0$, (3.1) and Lemma 1 [10], we find that there is $\gamma_0 > 0$ such that

$$\gamma_0 \int_{\Omega_n'} |\nabla w|^p \, dx \leqslant \|c\|_{\infty,\Omega} \left(\frac{a_0}{a_0-1} \right)^{p-2} \int_{\Omega_n'} |\nabla w|^{p-1} w + \left(a_0^{p-2}-1\right) \|c\|_{\infty,\Omega} \int_{\Omega_n'} |\nabla u|^{p-1} w$$
$$+ \left(a_0^{p-2}+a_0^{p-1}-2\right) \|c\|_{\infty,\Omega} \int_{\Omega_n'} |\nabla v|^{p-1} w - \int_{\Omega_n'} \alpha(\lambda_n) w.$$

We deduce, from $\alpha(\lambda_{n_0}) \leqslant \alpha(\lambda_n)$, that

$$\gamma_0 \int_{\Omega_n'} |\nabla w|^p \, dx \leqslant \int_{\Omega_n'} \Phi w + c_{a_0} \left(\int_{\Omega_n'} |\nabla w|^p \right)^{\frac{p-1}{p}} \left(\int_{\Omega_n'} w^p \right)^{\frac{1}{p}},$$

where $c_{a_0} = \|c\|_{\infty,\Omega} (\frac{a_0}{a_0-1})^{p-2}$.

In view of $\Omega_n \subset \Omega_{n_0}$ and (3.2), we get that

$$\gamma_0 \left(\int_{\Omega_n'} |\nabla w|^p \, dx \right)^{\frac{1}{p}} \leqslant c_{a_0} \left(\int_{\Omega_n'} w^p \right)^{\frac{1}{p}}. \tag{3.3}$$

Choose $\frac{Np}{N+p} < \theta < \min\{N, p\}$. By Sobolev's theorem, we have

$$\left(\int_{\Omega} w^{\theta^*} \, dx \right)^{\frac{1}{\theta^*}} \leqslant C \left(\int_{\Omega} |\nabla w|^{\theta} \, dx \right)^{\frac{1}{\theta}},$$

then

$$\left(\int_{\Omega_n'} w^{\theta^*} \, dx \right)^{\frac{1}{\theta^*}} \leqslant C \left(\int_{\Omega_n'} |\nabla w|^{\theta} \, dx \right)^{\frac{1}{\theta}};$$

using this inequality and Holder's inequality, we obtain

$$\left(\int_{\Omega_n'} w^{\theta^*} \, dx \right)^{\frac{1}{\theta^*}} \leqslant C \left(\int_{\Omega_n'} |\nabla w|^{p} \, dx \right)^{\frac{1}{p}} \left|\Omega_n'\right|^{\frac{1}{\theta}-\frac{1}{p}},$$

and from (3.3), we have

$$\left(\int_{\Omega_n'} w^{\theta^*} \, dx \right)^{\frac{1}{\theta^*}} \leqslant C \left(\int_{\Omega_n'} |w|^{p} \right)^{\frac{1}{p}} \left|\Omega_n'\right|^{\frac{1}{\theta}-\frac{1}{p}},$$

therefore

$$\left(\int_{\Omega_n'} w^{\theta^*} \, dx \right)^{\frac{1}{\theta^*}} \leqslant C \left(\int_{\Omega_n'} |w|^{\theta^*} \right)^{\frac{1}{\theta^*}} \left|\Omega_n'\right|^{\frac{1}{\theta}-\frac{1}{\theta^*}}.$$

This gives

$$1 \leqslant C\left|\Omega_n'\right|^{\frac{1}{\theta}-\frac{1}{\theta^*}} \to 0, \quad \text{as } n \to \infty,$$

which is impossible. □

Lemma 3.2. *Let $u \in W^{1,p}(\Omega)$ be a function which satisfies $L(u) = 0$, in Ω. If $u \in L^\infty_{loc}(\Omega)$, then $|\nabla u| \in L^\infty_{loc}(\Omega)$.*

Proof. Let G be a compact subset of Ω. There is a sub-domain Ω' such that $G \subset \Omega' \subset \overline{\Omega}' \subset \Omega$. Then, $G \subset \bigcup_{x\in G} B(x, r_x) \subset \Omega'$, $r_x \in (0,1)$. So, $G \subset \bigcup_{x\in G} B(x, \frac{r_x}{2}) \subset \Omega'$, with $\overline{B}(x, \frac{r_x}{2}) \subset \Omega'$. Since G is compact, we have that there is $B(x_i, r_i)$, $i = 1, \dots, k'$, with $r_i \in (0,1)$ such that $G \subseteq \bigcup B(x_i, r_i) \subseteq \Omega'$. Now, we put (see [17])

$$h(x,t,\eta) = \begin{cases} |\eta|^{p-1} + F(x,t) & \text{if } t \leqslant \|u\|_{\infty,\overline{\Omega}'}, \\ |\eta|^{p-1} + F(x, \|u\|_{\infty,\overline{\Omega}'}) & \text{if } t > \|u\|_{\infty,\overline{\Omega}'}. \end{cases}$$

Hence

$$\left|h(x,t,\eta)\right| \leqslant \left[\|c\|_{\infty,\Omega'} + M'\right]\left(1+|\eta|\right)^p$$

with

$$M' = \sup_{x\in\overline{\Omega}'} F\left(x, \|u\|_{\infty,\overline{\Omega}'}\right).$$

Notice that

$$\Delta_p(u) = h(x,u,\nabla u) \quad \text{in } \Omega'.$$

Then, Theorem 2.2 implies that $|\nabla u| < c_i$ in $B(x_i, r_i)$. Therefore,

$$|\nabla u| < \max\left\{c_i / i = 1, \dots, k'\right\} \quad \text{in } G. \qquad \square$$

Now, we consider the problem

$$(P_k) \quad \begin{cases} -\Delta_p(u) + c(x)|\nabla u|^{p-1} + F(x,u) = 0, \\ u/\partial\Omega = k. \end{cases}$$

Theorem 3.3. *Assume that there is $q > p-1$ such that $\Lambda_1(x) := \inf_{u>0} \frac{F(x,u)}{u^q} > 0$ and $\inf \Lambda_1 > 0$. Then, the problem (P_k) has a weak solution u, which satisfies*

$$u \leqslant M e^{\frac{1}{v_0}}$$

with

$$M := \left(\frac{\lambda_\infty}{\inf \Lambda_1}\left(\frac{2p}{(q-p+1)e}\right)^{2p}\right)^{\frac{1}{q-p+1}}.$$

Proof. Let Ω_n be a sub-smooth-domain which satisfies $\Omega_n \subset \overline{\Omega}_n \subset \Omega$ and $d(\partial\Omega_n, \partial\Omega) \to 0$, as $n \to \infty$. We consider the problem

$$\left(P_k^n\right) \quad \begin{cases} -\Delta_p(u) + c(x)|\nabla u|^{p-1} + F(x,u) = 0, \\ u/\partial\Omega_n = k. \end{cases}$$

Putting $v = u - k$, we see that v satisfies

$$\left(P_n'\right) \quad \begin{cases} -\Delta_p(v) + c(x)|\nabla v|^{p-1} + F(x, v+k) = 0, \\ v/\partial\Omega_n = 0. \end{cases}$$

We recall that v_0 is a solution of the problem (2.2) and we put $w_2 := M e^{\frac{1}{v_0}}$. Using the fact that

$$s^{2p} e^{\frac{q-p+1}{s}} \geqslant \left(\frac{q-p+1}{2p} e\right)^{2p}, \quad \forall s \in \mathbb{R}^+,$$

we obtain

$$-\Delta_p(w_2) + c|\nabla w_2|^{p-1} + F(x, w_2) \geqslant 0, \tag{3.4}$$

and easily we remark

$$-\Delta_p(-k) + c\left|\nabla(-k)\right|^{p-1} + F(x, -k+k) \leqslant 0. \tag{3.5}$$

By using the fact that $w_2 = \infty$ on $\partial\Omega$, we get $w_2 - k \geqslant 0$ on $\partial\Omega_n$, for n sufficiently large. Since $\nabla w_2 \in Ł^\infty(\Omega_n)$, we deduce that the solutions $w_2 - k$ and $-k$ are, respectively, super-solution and sub-solution of the problem (P'_n). On the other hand, we take

$$h(x, u, \nabla u) = c(x)|\nabla u|^{p-1} + F(x, u+k)$$

then

$$\left|h(x, u, \nabla u)\right| \leqslant K(x) + \|c\|_{\infty,\Omega_n}|\nabla u|^{p-1},$$

where $-k < u < w_2 - k$ and $K(x) = F(x, \|w_2\|_{\infty,\Omega_n})$. From the continuity of the function $F(., \|w_2\|_{\infty,\Omega_n})$ in Ω, we have $K \in L^t(\Omega_n)$ for all $t > p^*/p'$. In view of Theorem 2.1 we deduce that the problem (P'_n) admits a solution between $-k$ and $w_2 - k$. Thus, the problem (P_k^n) admits a solution u_n such that $0 \leqslant u_n \leqslant w_2$. Now, we define

$$v_n = \begin{cases} u_n & \text{in } \Omega_n, \\ k & \text{in } \Omega \setminus \Omega_n. \end{cases}$$

We remark that $L(k) \geqslant L(u_n) = 0$, in Ω_n and $u_n = k$ on $\partial\Omega_n$. In view of Lemma 3.2, $|\nabla u_n| \in L^\infty_{loc}(\Omega_n)$. Then, the maximum principle gives $u_n \leqslant k$, for all large $n \in \mathbb{N}$. Thus, $u_{n+1} \leqslant k$. Since $L(u_{n+1}) = L(u_n) = 0$ in Ω_n and $u_n = k \geqslant u_{n+1}$ on $\partial\Omega_n$, by using again the maximum principle, we obtain $u_{n+1} \leqslant u_n$. Thereby, we deduce that v_n is decreasing and since $0 \leqslant v_n \leqslant \max\{k, w_2\}$, it follows that there is a function w such that $v_n \to w$.

Fixing $D \subset \overline{D} \subset \Omega$, there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, $D \subset \Omega_{n_0} \subset \Omega_n$. Define the function test as $\psi = v_n\theta^p$ with $\theta = 1$ in D, $0 < \theta < 1$ and $\theta \in C_c^\infty(\Omega_{n_0})$. Then

$$\begin{aligned}\left\langle \Delta_p(v_n), \psi\right\rangle &= -\int_{\Omega_{n_0}} |\nabla v_n|^p\theta^p\,dx - p\int_{\Omega_{n_0}} |\nabla v_n|^{p-2}\theta^{p-1}v_n\nabla v_n\nabla\theta\,dx \\ &= -\int_{\Omega_{n_0}} |\nabla u_n|^p\theta^p\,dx - p\int_{\Omega_{n_0}} |\nabla u_n|^{p-2}\theta^{p-1}u_n\nabla u_n\nabla\theta\,dx.\end{aligned}$$

Thanks to $\langle\Delta_p(v_n), \psi\rangle = \langle\Delta_p(u_n), \psi\rangle$, $0 \leqslant u_n \leqslant w_2$ and $F(x, .)$ is non-decreasing, we deduce that

$$\begin{aligned}\int_{\Omega_{n_0}} |\nabla u_n|^p\theta^p\,dx &\leqslant p\|w_2\|_{\infty,\Omega_{n_0}}\|\nabla\theta\|_\infty\int_{\Omega_{n_0}} |\nabla u_n|^{p-1}\theta^{p-1}\,dx + \|w_2\|_{\infty,\Omega_{n_0}}\int_{\Omega_{n_0}} \theta^{p-1}\theta|c||\nabla u_n|^{p-1}\,dx \\ &\quad + \|w_2\|_{\infty,\Omega_{n_0}}\int_{\Omega_{n_0}} \theta F(x, w_2)\,dx \\ &\leqslant \|w_2\|_{\infty,\Omega_{n_0}}\left[p\|\nabla\theta\|_\infty + \|c\|_{\infty,\Omega_{n_0}}\right]\int_{\Omega_{n_0}} |\nabla u_n|^{p-1}\theta^{p-1}\,dx + \|w_2\|_{\infty,\Omega_{n_0}}\int_{\Omega_{n_0}} F(x, w_2)\,dx \\ &\leqslant |\Omega_{n_0}|^{\frac{1}{p}}\|w_2\|_{\infty,\Omega_{n_0}}\left[p\|\nabla\theta\|_\infty + \|c\|_{\infty,\Omega_{n_0}}\right]\left(\int_{\Omega_{n_0}} |\nabla u_n|^p\theta^p\,dx\right)^{\frac{p-1}{p}} + \|w_2\|_{\infty,\Omega_{n_0}}\int_{\Omega_{n_0}} F(x, w_2)\,dx.\end{aligned}$$

Then

$$\int_{\Omega_{n_0}} \theta^p|\nabla u_n|^p\,dx \leqslant C\left[\left(\int_{\Omega_{n_0}} |\nabla u_n|^p\theta^p\,dx\right)^{\frac{p-1}{p}} + \int_{\Omega_{n_0}} F(x, w_2)\right],$$

therefore,

$$\int_D |\nabla u_n|^p\,dx \leqslant C\left[\left(\int_{\Omega_{n_0}} |\nabla u_n|^p\theta^p\,dx\right)^{\frac{p-1}{p}} + \int_{\Omega_{n_0}} F(x, w_2)\right], \tag{3.6}$$

where $C = C(\|w_2\|_{\infty,\Omega_{n_0}}, p, \|c\|_{\infty,\Omega_{n_0}})$. Hence, $\int_{\Omega n_0} \theta|\nabla u_n|^p\,dx$ is bounded. It follows that $\int_D |\nabla u_n|^p\,dx$ is also bounded. Then $\nabla u_n \to g$ in $(L^p(D)^N)'$. Moreover, since $(w-u_n)_n$ is monotone and $(w-u_n) \to 0$,

$$\left|\int_D (u_n - w)\nabla\varphi\,dx\right| \leqslant \int_D (w-u_n)|\nabla\varphi|\,dx \to 0, \quad \forall\varphi \in C_c^\infty(D),$$

and hence

$$\int_D \nabla u_n.\varphi = -\int_D u_n.\nabla\varphi \quad \to \quad \int_D g.\varphi = -\int_D w.\nabla\varphi\,dx.$$

Consequently, $\nabla w = g$.

Letting $G \subseteq \overline{G} \subset \Omega$, there exist $\varphi \in C_0^\infty(\Omega)$ and $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $G \subset s_\varphi = \overline{\{x \in \Omega,\ \varphi(x) \neq 0\}} \subset \Omega_n$. Set

$$N_1 = \int_{s_\varphi} c|\nabla u_n|^{p-1}(u_n - w)\varphi, \qquad N_2 = \int_{s_\varphi} F(x, u_n)(u_n - w)\varphi,$$

$$N_3 = \int_{s_\varphi} |\nabla w|^{p-2}\nabla w\varphi\nabla(u_n - w), \qquad N_4 = \int_{s_\varphi} |\nabla w|^{p-2}(u_n - w)\nabla w\nabla\varphi$$

and

$$N_5 = \int_{s_\varphi} \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla w|^{p-2}\nabla w\big)(u_n - w)\nabla\varphi.$$

Hence

$$\int_{s_\varphi} \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla w|^{p-2}\nabla w\big)\varphi\nabla(u_n - w) = -N_1 - N_2 - N_3 - N_4 - N_5.$$

Letting $\psi = \varphi|\nabla w|^{p-2}\nabla w$ be a test function, since $(\nabla u_n)_n$ converges weakly to ∇w, we get

$$N_3 = \int_{s_\varphi} \psi(\nabla u_n - \nabla w) \to 0.$$

Note that $(u_n)_n$ is monotone, hence $(|u_n - w|)_n = (w - u_n)_n$ is monotone. Thus, since $w - u_n \to 0$ and from (3.6) (with $D = S_\varphi$), we have that $\int_{S_\varphi} |\nabla u_n|^p$ is bounded. It follows that

$$|N_1| \leqslant \|c\|_{\infty,s_\varphi}\bigg(\int_{S_\varphi} |\nabla u_n|^p\bigg)^{\frac{p-1}{p}}\bigg(\int_{S_\varphi} (w-u_n)^p|\varphi|^p\bigg)^{\frac{1}{p}} \to 0,$$

$$|N_5| \leqslant \bigg[\bigg(\int_{S_\varphi} |\nabla u_n|^p\bigg)^{\frac{p-1}{p}} + \bigg(\int_{S_\varphi} |\nabla w|^p\bigg)^{\frac{p-1}{p}}\bigg]\bigg(\int_{S_\varphi} (w-u_n)^p|\nabla\varphi|^p\bigg)^{\frac{1}{p}} \to 0,$$

$$|N_2| \leqslant \big\|F\big(.,\|w_2\|_{\infty,S_\varphi}\big)\big\|_{\infty,S_\varphi}\int_{S_\varphi} (w-u_n)|\varphi| \to 0$$

and

$$|N_4| \leqslant \bigg(\int_{S_\varphi} |\nabla w|^p\bigg)^{\frac{p-1}{p}}\bigg(\int_{S_\varphi} (w-u_n)^p|\nabla\varphi|^p\bigg)^{\frac{1}{p}} \to 0.$$

Letting n converge to ∞, we get

$$\int_{S_\varphi} \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla w|^{p-2}\nabla w\big)\varphi\nabla(u_n - w) \to 0, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{3.7}$$

From Lemma 1 in [10], we have

$$\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla w|^{p-2}\nabla w\right)(\nabla u_n - \nabla w) > \gamma_0|\nabla u_n - \nabla w|^p. \tag{3.8}$$

Choosing $\varphi = 1$ in G and $0 \leqslant \varphi \leqslant 1$, we obtain

$$\int\limits_{s_\varphi} |\nabla u_n - \nabla w|^p \varphi \to 0.$$

This implies that $\int_G |\nabla u_n - \nabla w|^p \to 0$. Then, we deduce that there exist $h \in L^p(G)$ and a sub-sequence $(|\nabla u_n - \nabla w|)_n$ such that $h \geqslant |\nabla u_n - \nabla w|$ and $|\nabla u_n - \nabla w| \to 0$, a.e. in G. Then $\nabla u_n \to \nabla w$ a.e. in G. Hence,

$$|\nabla u_n|^{p-2}\nabla u_n \to |\nabla w|^{p-2}\nabla w$$

and

$$c|\nabla u_n|^{p-1} \to c|\nabla w|^{p-1}$$

a.e. in G. Since $(h(x) + |\nabla w|)^{p-1} \geqslant |\nabla u_n|^{p-1}$ and $(h(.) + |\nabla w|)^{p-1} \in L(G)$, the dominated convergence theorem gives

$$\int\limits_\Omega |\nabla u_n|^{p-2}\nabla u_n \rho \to \int\limits_\Omega |\nabla w|^{p-2}\nabla w \rho, \quad \forall \rho \in \left(C_0^\infty(G)\right)^N, \tag{3.9}$$

and

$$\int\limits_\Omega c|\nabla u_n|^{p-1}\rho \to \int\limits_\Omega c|\nabla w|^{p-1}\rho, \quad \forall \rho \in \left(C_0^\infty(G)\right)^N. \tag{3.10}$$

In addition, in view of the monotonic of u_n and since $u \to F(x,u)$ is non-decreasing, we obtain that $F(x,u_n) \to F(x,w)$. Using the fact that $|F(x,u_n)| \leqslant \|F(.,\|w_2\|_{\infty,s_\varphi})\|_{\infty,s_\varphi}$, we obtain that

$$\int\limits_\Omega F(x,u_n)\varphi \to \int\limits_\Omega F(x,w)\varphi, \tag{3.11}$$

and finally, from (3.9), (3.10) and (3.11) we obtain that w is a solution of the problem (P_k). $\square$

Theorem 3.4. *Make the same assumptions as in Theorem* 3.3. *Then the problem* (1.1) *admits a weak blow-up solution, which verifies*

$$u \leqslant \left(\frac{\lambda_\infty}{\inf \Lambda_1}\left(\frac{2p}{(q-p+1)e}\right)^{2p}\right)^{\frac{1}{q-p+1}} e^{\frac{1}{\nu_0}}.$$

Proof. Let $u_k \in W^{1,p}(\Omega)$ be a weak solution of (P_k). In view of (3.4), w_2 is a weak super-explosive solution of (P). From the maximum principle, $0 \leqslant u_k < u_{k+1} \leqslant w_2$. We thus get $u_k \to u$ with $0 \leqslant u \leqslant w_2$. Let us consider a sub-domain G such that $G \subseteq \overline{G} \subset \Omega$. From the fact that $(u_k)_k$, $F(x,.)$ are non-decreasing, we have

$$\int\limits_G F(x,u_k)\varphi\,dx \to \int\limits_G F(x,u)\varphi\,dx, \quad \forall \varphi \in D(\Omega).$$

Consider a function $\varphi \in D(\Omega)$, with $\overline{G} \subseteq S_\varphi = \overline{\{\varphi \neq 0\}}$, $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ in $\overline{G}$. Take φu_n as a test function. Then,

$$0 \leqslant \left\langle \Delta_p u_n - c(x)|\nabla u_n|^{p-1}, \varphi u_n\right\rangle = -\int\limits_{S_\varphi} |\nabla u_n|^p \varphi\,dx + \gamma_n,$$

with $\gamma_n = -\int_{S_\varphi} |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi u_n - \int_{S_\varphi} c(x)|\nabla u_n|^{p-1}\varphi u_n$. Therefore, by $0 \leqslant u_n \leqslant w_2$, we have that

$$\int\limits_{S_\varphi} |\nabla u_n|^p\varphi\,dx \leqslant \left(\int\limits_{S_\varphi} |\nabla u_n|^p\varphi\right)^{\frac{p-1}{p}} \left[p\|w_2\|_{\infty,S_\varphi}\left(\int\limits_{S_\varphi}\left|\nabla\left(\varphi^{\frac{1}{p}}\right)\right|^p\right)^{\frac{1}{p}} + \left\|cw_2\varphi^{\frac{1}{p}}\right\|_{\infty,S_\varphi}|S_\varphi|^{\frac{1}{p}}\right],$$

hence

$$\left(\int\limits_{S_\varphi} |\nabla u_n|^p\varphi\,dx\right)^{\frac{1}{p}} \leqslant p\|w_2\|_{\infty,S_\varphi}\left(\int\limits_{S_\varphi}\left|\nabla\left(\varphi^{\frac{1}{p}}\right)\right|^p\right)^{\frac{1}{p}} + \left\|cw_2\varphi^{\frac{1}{p}}\right\|_{\infty,S_\varphi}|S_\varphi|^{\frac{1}{p}}.$$

Consequently, $\int_{S_\varphi} |\nabla u_n|^p \varphi \, dx$ is bounded. It follows that $\int_G |\nabla u_n|^p \, dx$ is also bounded. Thereby, we can choose a subsequence, still denoted by ∇u_k, such that $\nabla u_k \to g$ in $L^p(G)'$. The remainder of the proof is the same as that in the proof of Theorem 3.3. Indeed, since u_k is non-decreasing and $u_k \to u$, we get

$$\left| \int\limits_\Omega (u_k - u)\nabla\varphi \right| < \int\limits_\Omega (u - u_k)|\nabla\varphi| \to 0$$

for all $\varphi \in D(G)$. Thus

$$\int\limits_\Omega \nabla u_k.\varphi = \int\limits_\Omega u_k.\nabla\varphi \to \int\limits_\Omega g.\varphi = -\int\limits_\Omega u.\nabla\varphi,$$

and we deduce that $\nabla u = g$. From (3.9) and (3.10), we find that

$$\int\limits_\Omega |\nabla u_n|^{p-2}\nabla u_n \rho \to \int\limits_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi, \quad \forall\varphi \in D(\Omega), \tag{3.12}$$

and

$$\int\limits_\Omega c|\nabla u_n|^{p-1}\rho \to \int\limits_\Omega c|\nabla u|^{p-1}\varphi, \quad \forall\rho \in D(\Omega). \tag{3.13}$$

Finally, u is a weak explosive solution of the problem (1.1). $\square$

Theorem 3.5. *Assume that there is $q \leqslant p-1$ such that $\Lambda_2(x) := \sup_{u>0} \frac{F(x,u)}{u^q} \in L^\infty(\Omega)$. The problem* (1.1) *has no blow-up solution.*

Proof. We proceed by contradiction. Suppose that there exists a positive solution of (1.1), denoted by w. We define $\bar v(x) = \ln(1+w)$ (see [21]). Then, an easy computation gives

$$\Delta_p \bar v \leqslant -(p-1)\left(\frac{|\nabla w|}{(1+w)}\right)^p + \|c\|_{\infty,\Omega}\left(\frac{|\nabla w|}{(1+w)}\right)^{p-1} + \|\Lambda_2\|_{\infty,\Omega}\frac{w^q}{(1+w)^{p-1}}.$$

Since $q \leqslant p-1$ and $p > p-1$, we can find two constants K_1 and K_2 such that

$$-(p-1)\left(\frac{|\nabla w|}{(1+w)}\right)^p + \|c\|_{\infty,\Omega}\left(\frac{|\nabla w|}{(1+w)}\right)^{p-1} \leqslant K_1$$

and

$$\frac{w^q}{(1+w)^{p-1}} \leqslant K_2,$$

hence

$$\Delta_p \bar v \leqslant K_1 + K_2\|\Lambda_2\|_{\infty,\Omega}.$$

Let Ω_n be a sub-domain of Ω such that $\Omega_n \subset \overline{\Omega}_n \subset \Omega$ and $d(\partial\Omega_n, \partial\Omega) \to 0$. Given $A > K_1 + K_2\|\Lambda_2\|_{\infty,\Omega}$, the function $\underline{v} = \frac{(p-1)}{p}(\frac{A}{N})^{\frac{1}{p-1}}|x|^{p/(p-1)}$ is a solution of $\Delta_p u = A$ in $\mathbb{R}^N$ (see [10] and the references therein). Therefore, $-\Delta_p \bar v \geqslant -\Delta_p \underline{v}$. Using the fact that $w(x) \to +\infty$, as $d(x, \partial\Omega) \to 0$, we obtain that there exits n_0 such that $\bar v > \|\underline{v}\|_{\infty,\Omega}$, on $\partial\Omega_n$, as soon as $n > n_0$. Thereby, according to the maximum principle we obtain

$$\bar v(x) = \ln\bigl(1 + w(x)\bigr) \geqslant \frac{(p-1)}{p}\left(\frac{A}{N}\right)^{\frac{1}{p-1}}|x|^{p/(p-1)} \quad \text{in } \Omega_n$$

as soon as $n > n_0$. Consequently,

$$\bar v(x) = \ln\bigl(1 + w(x)\bigr) \geqslant \frac{(p-1)}{p}\left(\frac{A}{N}\right)^{\frac{1}{p-1}}|x|^{p/(p-1)} \quad \text{in } \Omega.$$

Letting $A \to \infty$, we deduce $\bar v = \infty$, which is a contradiction. $\square$

Theorem 3.6. *Suppose that there exist a sub- and super-solution of the problem* (1.1) *satisfying $\bar v(x) - \underline{v}(x) \to 0$, as $d(x, \partial\Omega) \to 0$. Then, problem* (1.1) *has at most a positive solution.*

Proof. Suppose that (1.1) has two nonnegative blow-up solutions u_1 and u_2 such that $u_1 \neq u_2$. Let Ω_n be a sub-smooth-domain which satisfies $\Omega_n \subset \overline{\Omega}_n \subset \Omega$ and $d(\partial\Omega_n, \partial\Omega) \to 0$, as $n \to \infty$. Consider $D \subset \Omega$. Given $\varepsilon > 0$. Since $\overline{v}(x) - \underline{v}(x) \to 0$, as $d(x, \partial\Omega) \to 0$, and $\underline{v} \leqslant u_i \leqslant \overline{v}$, $i = 1, 2$, we have $u_1(x) - u_2(x) \to 0$, as $d(x, \partial\Omega) \to 0$. Then, there is $n \in \mathbb{N}$ such that $D \subset \Omega_n$, $u_2(x) < u_1(x) + \varepsilon$ and $u_1(x) < u_2(x) + \varepsilon$, on $\partial\Omega_n$. It is clear that $L(u_1) = L(u_2) = 0$ in Ω_n. Thus by the maximum principle we deduce that $u_1 \leqslant u_2 + \varepsilon$ in Ω_n. By the same way we show that $u_2 \leqslant u_1 + \varepsilon$, in Ω_n. Therefore, $u_1 \leqslant u_2 + \varepsilon$ and $u_1 \leqslant u_2 + \varepsilon$ in D. Letting $\varepsilon \to 0$ gives $u_1 = u_2$. $\square$

4. Examples

Now, we introduce some examples.

Example 1. In this example we establish weak solutions of the singular boundary value problem

$$\begin{cases} \Delta_p u = \Lambda(x) u^q & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

with $q > 1$, $\Lambda \in C(\overline{\Omega})$ and $\Lambda > 0$.

Theorem 4.1. *Problem* (4.1) *has a positive solution if and only if* $q > p - 1$.

Proof. From Theorems 3.3 and 3.5, the problem (4.1) has a positive solution if and only if $q > p - 1$.

Define

$$M = \left\{ \frac{1}{\inf \Lambda} \left(\frac{p}{q - p + 1} \right)^{p-1} \left\| \frac{q+1}{q-p+1} (p-1) |\nabla v_0|^p + v_0 \right\|_{\infty, \Omega} \right\}^{\frac{1}{q-p+1}},$$

and

$$m = \left\{ \left(\frac{p-1}{q-p+1} \right)^{p-1} \inf_{x \in \Omega} \left\{ \frac{q}{q-p+1} (p-1) \frac{|\nabla v_0|^{p-1}}{v_0} + 1 \right\} \frac{1}{\sup \Lambda} \right\}^{\frac{1}{q-p+1}}.$$

Let $\overline{v} = M v_0^{-\frac{p}{q-p+1}}$ and $\underline{v} = m v_0^{-\frac{p-1}{q-p+1}}$. We thus get

$$\Delta_p \overline{v} \leqslant \Lambda(x) \overline{v}^q \quad \text{in } \Omega$$

and

$$\Delta_p \underline{v} \geqslant \Lambda(x) \underline{v}^q \quad \text{in } \Omega.$$

According to Theorem 3.1, we have $\underline{v} \leqslant u \leqslant \overline{v}$. $\square$

Example 2. In this second example, we will consider a function F which is not a function with separate variables. In fact, the example is as follows:

$$\begin{cases} \Delta_p u + \big|c(x)\big| |\nabla u|^{p-1} = \dfrac{u + |x| + 2}{u + |x| + 1} f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where f is a non-decreasing function satisfying

$$\Lambda_1 = \inf_{u \geqslant 0} f(u)/u^q > 0, \qquad \Lambda_2 = \sup_{u \geqslant 0} f(u)/u^q < \infty.$$

Theorem 4.2. *Problem* (4.2) *has a positive solution if and only if* $q > p - 1$.

Proof. Let

$$M = \left\{ \frac{1}{\Lambda_1} \left(\frac{p}{q-p+1} \right)^{p-1} \left\| \frac{q+1}{q-p+1} (p-1) |\nabla v_0|^p + v_0 + |c| w |\nabla w|^{p-1} \right\|_{\infty, \Omega} \right\}^{\frac{1}{q-p+1}},$$

and

$$m = \left\{ \frac{1}{2\Lambda_2} \left(\frac{p-1}{q-p+1} \right)^{p-1} \inf_{x \in \Omega} \left\{ \frac{q}{q-p+1} (p-1) \frac{|\nabla v_0|^p}{v_0} + 1 + |c| |\nabla v_0|^{p-1} \right\} \right\}^{\frac{1}{q-p+1}}.$$

An easy computation shows that $\bar{v} = Mv_0^{-\frac{p}{q-p+1}}$ and $\underline{v} = mv_0^{-\frac{p-1}{q-p+1}}$ are a sub-solution and super-solution of the problem (4.2), respectively. By using again Theorems 3.3 and 3.5, we obtain the result. □

Remark 4.3. Let us note that we can have v_0 explicitly if $\Omega = B(0, R)$. Indeed, we know that the radial solutions of the problem $\Delta_p v_0 = 1$ in the ball $B(0; R)$ and $v_0 = 0$ on $\partial B(0, R)$, satisfy the differential equation

$$\begin{cases} (|u'|^{p-2}u')' + \dfrac{N-1}{r}|u'|^{p-2}u' = 1, \\ u(R) = 0, \end{cases} \tag{4.3}$$

with $r = |x|$. Thus, we obtain

$$v_0(x) = u(r) = \left(\frac{1}{N+1}\right)^{p-1} \frac{p-1}{p}\left(R^{\frac{p}{p-1}} - |x|^{\frac{p}{p-1}}\right).$$

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References

[1] J.B. Keller, On solutions of $\Delta u = f(u)$, Comm. Pure Appl. Math. 10 (1957) 503–510.
[2] R. Osserman, On the inequality $\Delta u \geqslant f(u)$, Pacific J. Math. 7 (1957) 1641–1647.
[3] C. Bandle, M. Marcus, Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour, J. Anal. Math. 58 (1992) 9–24.
[4] C. Bandle, M. Marcus, On second order effects in the boundary behaviour of large solutions of semilinear elliptic problems, Differential Integral Equations 11 (1998) 23–34.
[5] C. Bandle, M. Marcus, Asymptotic behaviour of solutions and their derivatives for semilinear elliptic problems with blow-up on the boundary, Ann. Inst. H. Poincare 12 (1995) 155–171.
[6] C. Bandle, Y. Cheng, G. Porru, Boundary blow-up in semilinear elliptic problems with singular weights at the boundary, Institut Mittag-Leffler, Report No. 39, 1999/2000, pp. 1–14.
[7] E. DiBenedetto, C^{1+a} local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (8) (1983) 827–850.
[8] M. Guedda, L. Véron, Quasilinear elliptic equations involving critical Sobolev exponents, Nonlinear Anal. 13 (8) (1989) 879–902.
[9] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (11) (1988) 1203–1219.
[10] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations 51 (1984) 126–150.
[11] M. Del Pino, R. Letelier, The influence of domain geometry in boundary blow-up elliptic problems, Nonlinear Anal. 48 (6) (2002) 897–904.
[12] G. Diaz, R. Letelier, Explosive solutions of quasilinear elliptic equations: Existence and uniqueness, Nonlinear Anal. 20 (1993) 97–125.
[13] J. Garcia-Melian, R. Letelier-Albornoz, J. Sabina de Lis, Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up, Proc. Amer. Math. Soc. 129 (12) (2001) 3593–3602.
[14] V.A. Kondrat'ev, V.A. Nikishkin, Asymptotics, near the boundary, of a solution of a singular boundary value problem for a semilinear elliptic equation, Differential Equations 26 (1990) 345–348.
[15] C. Loewner, L. Nirenberg, Partial differential equations invariant under conformal of projective transformations, in: Contributions to Analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 1974, pp. 245–272.
[16] L. Véron, Semilinear elliptic equations with uniform blow-up on the boundary, J. Anal. Math. 59 (1992) 231–250.
[17] A. Mohammed, Boundary behavior of blow-up solutions to some weighted non-linear differential equations, Electron. J. Differential Equations 78 (2002) 1–15.
[18] A. Mohammed, G. Porcu, G. Porru, Large solutions to some non-linear O.D.E. with singular coefficients, Nonlinear Anal. 47 (1) (2001) 513–524.
[19] M.C. Leon, Existence result for quasi-linear problems via ordered sub and supersolution, Ann. Fac. Sci. Toulouse Math. (6) 6 (4) (1997) 591–608.
[20] S. Kim, A note on boundary blow-up problem of $\Delta u = u^p$, IMA preprint No. 1820, 2002.
[21] M. Ghergu, V. Rădulescu, Nonradial blow up solutions of sublinear elliptic equations with gradient term, Commun. Pure Appl. Anal. 3 (3) (2004).
[22] M. Ghergu, V. Rădulescu, Explosive solutions of semilinear elliptic systems with gradient term, Rev. R. Acad. Cienc. Ser. A Mat. 97 (3) (2003) 467–475.
[23] M. Ghergu, V. Rădulescu, Singular Elliptic Problems. Bifurcation and Asymptotic Analysis, Oxford Lecture Ser. Math. Appl., vol. 37, Oxford University Press, 2008.
[24] A.V. Lair, A necessary and sufficient condition for existence of large solutions to semilinear elliptic equations, J. Math. Anal. Appl. 240 (1999) 205–218.
[25] J. Melián, Boundary behavior for large solutions to elliptic equations with singular weights, Nonlinear Anal. 67 (2007) 818–826.
[26] J. Melián, A remark on the existence of large solutions via sub and supersolutions, Electron. J. Differential Equations 110 (2003) 1–4.
[27] J.V. Goncalves, A. Roncalli, Existence, non-existence and asymptotic behavior of blow-up entire solutions of semilinear elliptic equations, J. Math. Anal. Appl. 321 (2006) 524–536.
[28] T. Ouyang, Z. Xie, The uniqueness of blow-up solution for radially symmetric semilinear elliptic equation, Nonlinear Anal. 64 (2006) 2129–2142.
[29] Zongming Guoa, Junli Shangb, Remarks on uniqueness of boundary blow-up solutions, Nonlinear Anal. 66 (2007) 484–497.
[30] Ahmed Mohammed, Boundary asymptotic and uniqueness of solutions to the p-Laplacian with infinite boundary values, J. Math. Anal. Appl. 325 (2007) 480–489.
[31] S. Huang, Q. Tian, Asymptotic behavior of large solutions to p-Laplacian of Bieberbach Rademacher type, Nonlinear Anal. 71 (2009) 5773–5780.